

Counting rational points of an algebraic variety over finite fields *

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Abstract. Let \mathbb{F}_q denote the finite field of odd characteristic p with q elements ($q = p^n, n \in \mathbb{N}$) and \mathbb{F}_q^* represent the nonzero elements of \mathbb{F}_q . In this paper, by using the Smith normal form we give an explicit formula for the number of rational points of the algebraic variety defined by the following system of equations over \mathbb{F}_q :

$$\left\{ \begin{array}{l} \sum_{i=1}^{r_1} a_{1i} x_1^{e_{i1}^{(1)}} \dots x_{n_1}^{e_{i,n_1}^{(1)}} + \sum_{i=r_1+1}^{r_2} a_{1i} x_1^{e_{i1}^{(1)}} \dots x_{n_2}^{e_{i,n_2}^{(1)}} - b_1 = 0, \\ \sum_{j=1}^{r_3} a_{2j} x_1^{e_{j1}^{(2)}} \dots x_{n_3}^{e_{j,n_3}^{(2)}} + \sum_{j=r_3+1}^{r_4} a_{2j} x_1^{e_{j1}^{(2)}} \dots x_{n_4}^{e_{j,n_4}^{(2)}} - b_2 = 0, \end{array} \right.$$

where the integers $1 \leq r_1 < r_2, 1 \leq r_3 < r_4, 1 \leq n_1 < n_2, 1 \leq n_3 < n_4, n_1 \leq n_3, b_1, b_2 \in \mathbb{F}_q, a_{1i} \in \mathbb{F}_q^* (1 \leq i \leq r_2), a_{2j} \in \mathbb{F}_q^* (1 \leq j \leq r_4)$ and the exponent of each variable is positive integer. An example is also presented to demonstrate the validity of the main result.

Keywords and phrases: Finite field, algebraic variety, rational point, Smith normal form, exponent matrix.

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1 Introduction

Let \mathbb{F}_q denote the finite field of q elements with odd characteristic p ($q = p^n, n \in \mathbb{N}$ (the set of positive integers)) and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let m be a positive integer, $f_i(x_1, \dots, x_n) (i = 1, \dots, m)$ be some polynomial with n variables over \mathbb{F}_q and V

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denote the algebraic variety defined by the following system of equations over \mathbb{F}_q :

$$\begin{cases} f_1(x_1, \dots, x_n) = 0, \\ \dots\dots\dots \\ f_m(x_1, \dots, x_n) = 0. \end{cases}$$

By $N_q(V)$ we denote the number of \mathbb{F}_q -rational points on the algebraic variety V in \mathbb{F}_q^n . That is,

$$N_q(V) := \#\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid f_i(x_1, \dots, x_n) = 0, i = 1, \dots, m\}.$$

Particularly, if $m = 1$, then we use $N_q(f)$ to stand for $N_q(V)$.

Studying the exact value of $N_q(V)$ is one of the main topics in finite fields. The degrees $\deg(f_i)$ play an important role in the estimate of $N_q(V)$. Let $\lceil x \rceil$ denote the least integer $\geq x$ and ord_q denote the additive valuation such that $\text{ord}_q q = 1$. In 1964, Ax [3] generalized the Chevalley-Warning theroem by showing that

$$\text{ord}_q N_q(V) \geq \left\lceil \frac{n - \sum_{i=1}^m \deg f_i}{\sum_{i=1}^m \deg f_i} \right\rceil.$$

Later on, further works were done by Katz [20], Adolphson-Sperber [1]-[2], Moreno-Moreno [22] and Wan [29]-[31].

It is difficult to give an explicit formula for $N_q(V)$ in general. Finding explicit formula for $N_q(f)$ under certain conditions has attracted many authors for many years (see, for instance, [18] et al). It is well known that there exists an explicit formula for $N_q(f)$ with $\deg(f) \leq 2$ in \mathbb{F}_q (see, for example, [19] and [21]). One first considered the diagonal hypersurface:

$$a_1 x_1^{e_1} + \dots + a_n x_n^{e_n} - b = 0, \quad 1 \leq i \leq n, \quad a_i \in \mathbb{F}_q^*, \quad b \in \mathbb{F}_q, \quad e_i > 0, \quad (1.1)$$

and much work has been done to seek for the number of rational points of the hypersurface (1.1)(see, for instance, [26] and [34]-[35]). Carlitz [6], Cohen [12] and Hodges [13] counted the rational points on the following k -linear hypersurface

$$a_1 x_{11} \dots x_{1k} + a_2 x_{21} \dots x_{2k} + \dots + a_n x_{n1} \dots x_{nk} - b = 0, \quad (1.2)$$

with $a_i \in \mathbb{F}_q^* (1 \leq i \leq n)$, $b \in \mathbb{F}_q$. Cao [9], Cao and Sun [10] [11] studied the rational points on the following more general diagonal hypersurface

$$a_1 x_{11}^{e_{11}} \dots x_{1n_1}^{e_{1n_1}} + a_2 x_{21}^{e_{21}} \dots x_{2n_2}^{e_{2n_2}} + \dots + a_r x_{r1}^{e_{r1}} \dots x_{rn_r}^{e_{rn_r}} = 0,$$

with $1 \leq i \leq r$, $1 \leq j \leq n_i$, $e_{ij} \in \mathbb{N}$, $a_i \in \mathbb{F}_q^*$. Clearly, this extends (1.1) and (1.2) when $b = 0$. On the other hand, Baoulina [4] [5], Pan, Zhao and Cao [23] considered the hypersurface of the form $(a_1 x_1^{m_1} + \dots + a_n x_n^{m_n})^\lambda - b x_1^{k_1} \dots x_n^{k_n} = 0$ which extended the results of Carlitz in [7] and [8].

If $f = a_1 x_1^{e_{11}} \dots x_n^{e_{1n}} + \dots + a_s x_1^{e_{s1}} \dots x_n^{e_{sn}} - b$ with $e_{ij} > 0$ and $a_i \in \mathbb{F}_q^*$ for $1 \leq i \leq s$ and $1 \leq j \leq n$, $b \in \mathbb{F}_q$, then a formula for $N_q(f)$ was given by Sun [27]. Moreover, if $s = n$ and $\gcd(\det(e_{ij}), q-1) = 1$ ($\det(e_{ij})$ represents the

determinant of the $n \times n$ matrix (e_{ij}) , then Sun [27] gave the explicit formula for the number of rational points as follows:

$$N(f) = \begin{cases} q^n - (q-1)^n + \frac{(q-1)^n + (-1)^n (q-1)}{q}, & \text{if } b = 0, \\ \frac{(q-1)^n - (-1)^n}{q}, & \text{if } b \neq 0. \end{cases}$$

Wang and Sun [32] gave the formula for the number of rational points of the following hypersurface

$$a_1 x_1^{e_{11}} + a_2 x_1^{e_{11}} x_2^{e_{22}} + \dots + a_n x_1^{e_{n1}} x_2^{e_{n2}} \dots x_n^{e_{nn}} - b = 0,$$

with $e_{ij} \geq 0$, $a_i \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$. In 2005, Wang and Sun [33] extended the results of [27] and [32]. Recently, Hu, Hong and Zhao [16] generalized Wang and Sun's results. In fact, they used the Smith normal form to present a formula for $N_q(f)$ with f being given by:

$$f = \sum_{j=0}^{t-1} \sum_{i=1}^{r_{j+1}-r_j} a_{r_j+i} x_1^{e_{r_j+i,1}} \dots x_{n_{j+1}}^{e_{r_j+i,n_{j+1}}} - b, \quad (1.3)$$

where the integers $t > 0$, $r_0 = 0 < r_1 < r_2 < \dots < r_t$, $1 \leq n_1 < n_2 < \dots < n_t$, $b \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q^*$ ($1 \leq i \leq r_t$) and the exponents e_{ij} of each variable are positive integers. On the other hand, Yang [36] followed Sun's method and gave a formula for the rational points $N_q(V)$ on the following algebraic variety V over \mathbb{F}_q :

$$\begin{cases} a_{11} x_1^{e_{11}^{(1)}} \dots x_n^{e_{1n}^{(1)}} + \dots + a_{1s} x_1^{e_{s1}^{(1)}} \dots x_n^{e_{sn}^{(1)}} - b_1 = 0, \\ \dots\dots\dots \\ a_{m1} x_1^{e_{11}^{(m)}} \dots x_n^{e_{1n}^{(m)}} + \dots + a_{ms} x_1^{e_{s1}^{(m)}} \dots x_n^{e_{sn}^{(m)}} - b_m = 0. \end{cases}$$

Song and Chen [25] continued to make use of Sun's method and obtained a formula for $N_q(V)$ with V being the algebraic variety over \mathbb{F}_q defined by:

$$\begin{cases} \sum_{j=1}^{s_1} a_{1j} x_1^{e_{j1}^{(1)}} \dots x_{n_1}^{e_{j,n_1}^{(1)}} + \sum_{j=s_1+1}^{s_2} a_{1j} x_1^{e_{j1}^{(1)}} \dots x_{n_2}^{e_{j,n_2}^{(1)}} - b_1 = 0, \\ \dots\dots\dots \\ \sum_{j=1}^{s_1} a_{mj} x_1^{e_{j1}^{(m)}} \dots x_{n_1}^{e_{j,n_1}^{(m)}} + \sum_{j=s_1+1}^{s_2} a_{mj} x_1^{e_{j1}^{(m)}} \dots x_{n_2}^{e_{j,n_2}^{(m)}} - b_m = 0. \end{cases}$$

Meanwhile, they proposed an interesting question which was recently answered by Hu and Hong [15]. A more general question was suggested by Hu, Hong and Zhao in [16] that can be stated as follows.

Problem 1.1. [16] Let $m \geq 1$ be an integer and t_1, \dots, t_m be positive integers. Find an explicit formula for the number of rational points on the following algebraic variety over \mathbb{F}_q :

$$\sum_{j=0}^{t_k-1} \sum_{i=1}^{r_{k,j+1}-r_{kj}} a_{r_{kj}+i} x_1^{e_{r_{kj}+i,1}^{(k)}} \dots x_{n_{k,j+1}}^{e_{r_{kj}+i,n_{k,j+1}}^{(k)}} - b_k = 0, \quad k = 1, \dots, m,$$

with $0 = r_{k0} < r_{k1} < r_{k2} < \dots < r_{k,t_k}$, $1 \leq n_{k1} < n_{k2} < \dots < n_{k,t_k}$ and $e_{r_{kj}+i,j}^{(k)} > 0$ being integers, $b_k \in \mathbb{F}_q$, $a_{r_{kj}+i} \in \mathbb{F}_q^*$ for $1 \leq k \leq m$, $1 \leq i \leq r_{k,t_k}$ and $1 \leq j \leq t_k$.

If $m = 1$, then the main result in [16] answers Problem 1.1. But it is kept open when $m \geq 2$. Clearly, Yang [36], Song and Chen [25] and Hu and Hong [15] gave a partial answer to Problem 1.1 when $m \geq 2$.

In what follows, we always let $r_i, n_i (i = 1, \dots, 4)$ be positive integers such that $1 \leq r_1 < r_2$, $1 \leq r_3 < r_4$, $1 \leq n_1 < n_2$, $1 \leq n_3 < n_4$, $n_1 \leq n_3$, $b_1, b_2 \in \mathbb{F}_q$, $a_{1i} \in \mathbb{F}_q^* (1 \leq i \leq r_2)$ and $a_{2j} \in \mathbb{F}_q^* (1 \leq j \leq r_4)$. Let $n = \max\{n_2, n_4\}$, $k = 1, 2$ and $f_k(x) := f_k(x_1, \dots, x_n) \in \mathbb{F}_q[x_1, \dots, x_n]$ be defined by

$$f_1(x) := f_1(x_1, \dots, x_n) = \sum_{i=1}^{r_2} a_{1i} x^{E_i^{(1)}} - b_1$$

and

$$f_2(x) := f_2(x_1, \dots, x_n) = \sum_{j=1}^{r_4} a_{2j} x^{E_j^{(2)}} - b_2. \quad (1.3)$$

with $E_i^{(1)} (1 \leq i \leq r_2)$ and $E_j^{(2)} (1 \leq j \leq r_4)$ being the vectors of non-negative integer components of dimension n defined as follows:

$$\left\{ \begin{array}{l} E_1^{(1)} = (e_{11}^{(1)}, \dots, e_{1n_1}^{(1)}, 0, \dots, 0), \quad x^{E_1^{(1)}} = x_1^{e_{11}^{(1)}} \dots x_{n_1}^{e_{1n_1}^{(1)}}, \\ \dots\dots\dots \\ E_{r_1}^{(1)} = (e_{r_1,1}^{(1)}, \dots, e_{r_1,n_1}^{(1)}, 0, \dots, 0), \quad x^{E_{r_1}^{(1)}} = x_1^{e_{r_1,1}^{(1)}} \dots x_{n_1}^{e_{r_1,n_1}^{(1)}}, \\ E_{r_1+1}^{(1)} = (e_{r_1+1,1}^{(1)}, \dots, e_{r_1+1,n_2}^{(1)}, 0, \dots, 0), \quad x^{E_{r_1+1}^{(1)}} = x_1^{e_{r_1+1,1}^{(1)}} \dots x_{n_2}^{e_{r_1+1,n_2}^{(1)}}, \\ \dots\dots\dots \\ E_{r_2}^{(1)} = (e_{r_2,1}^{(1)}, \dots, e_{r_2,n_2}^{(1)}, 0, \dots, 0), \quad x^{E_{r_2}^{(1)}} = x_1^{e_{r_2,1}^{(1)}} \dots x_{n_2}^{e_{r_2,n_2}^{(1)}}, \\ E_1^{(2)} = (e_{11}^{(2)}, \dots, e_{1n_3}^{(2)}, 0, \dots, 0), \quad x^{E_1^{(2)}} = x_1^{e_{11}^{(2)}} \dots x_{n_3}^{e_{1n_3}^{(2)}}, \\ \dots\dots\dots \\ E_{r_3}^{(2)} = (e_{r_3,1}^{(2)}, \dots, e_{r_3,n_3}^{(2)}, 0, \dots, 0), \quad x^{E_{r_3}^{(2)}} = x_1^{e_{r_3,1}^{(2)}} \dots x_{n_3}^{e_{r_3,n_3}^{(2)}}, \\ E_{r_3+1}^{(2)} = (e_{r_3+1,1}^{(2)}, \dots, e_{r_3+1,n_4}^{(2)}, 0, \dots, 0), \quad x^{E_{r_3+1}^{(2)}} = x_1^{e_{r_3+1,1}^{(2)}} \dots x_{n_4}^{e_{r_3+1,n_4}^{(2)}}, \\ \dots\dots\dots \\ E_{r_4}^{(2)} = (e_{r_4,1}^{(2)}, \dots, e_{r_4,n_4}^{(2)}, 0, \dots, 0), \quad x^{E_{r_4}^{(2)}} = x_1^{e_{r_4,1}^{(2)}} \dots x_{n_4}^{e_{r_4,n_4}^{(2)}}. \end{array} \right.$$

Then a special case of Problem 1.1 is the following question.

Problem 1.2. Let $f_1(x)$ and $f_2(x)$ be given as in (1.3). What is the formula for the number of rational points on the algebraic variety defined by the following system of equations over \mathbb{F}_q :

$$\begin{cases} f_1(x) = 0, \\ f_2(x) = 0? \end{cases} \quad (1.4)$$

When $n_1 = n_3$ and $n_2 = n_4$, this question was answered by Song and Chen [25]. However, if $n_1 \neq n_3$ or $n_2 \neq n_4$, then Problem 1.2 has not been solved yet.

In this paper, our main goal is to investigate Problem 1.2. We will follow and develop the method of [16] to study Problem 1.2. To state the main result, we first need to introduce some related concept and notation. Throughout this paper, we let

$$\begin{aligned} E_{11} &:= \begin{pmatrix} E_1^{(1)} \\ \vdots \\ E_{r_1}^{(1)} \end{pmatrix}_{r_1 \times m}, & E_{12} &:= \begin{pmatrix} E_{r_1+1}^{(1)} \\ \vdots \\ E_{r_2}^{(1)} \end{pmatrix}_{(r_2-r_1) \times m}, \\ E_{21} &:= \begin{pmatrix} E_1^{(2)} \\ \vdots \\ E_{r_3}^{(2)} \end{pmatrix}_{r_3 \times m} & \text{and } E_{22} &:= \begin{pmatrix} E_{r_3+1}^{(2)} \\ \vdots \\ E_{r_4}^{(2)} \end{pmatrix}_{(r_4-r_3) \times m}. \end{aligned}$$

For any integer l with $1 \leq l \leq 4$, we let $E^{(n_l)}$ denote the remaining of the exponent matrix of (1.4) which deletes the items containing the variable x_{n_l+1} . In fact, one has

$$\begin{aligned} E^{(n_1)} &:= E_{11} \text{ if } n_1 < n_3, \text{ and } \begin{pmatrix} E_{11} \\ E_{21} \end{pmatrix} \text{ if } n_1 = n_3, \\ E^{(n_3)} &:= \begin{pmatrix} E_{11} \\ E_{21} \end{pmatrix} \text{ if } n_3 < n_2, \text{ and } \begin{pmatrix} E_{11} \\ E_{12} \\ E_{21} \end{pmatrix} \text{ otherwise,} \\ E^{(n_4)} &:= \begin{pmatrix} E_{11} \\ E_{21} \\ E_{22} \end{pmatrix} \text{ if } n_4 < n_2, \text{ and } \begin{pmatrix} E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{pmatrix} \text{ otherwise.} \end{aligned}$$

If $n_3 \leq n_2$, then

$$E^{(n_2)} := \begin{pmatrix} E_{11} \\ E_{12} \\ E_{21} \end{pmatrix} \text{ if } n_2 < n_4, \text{ and } \begin{pmatrix} E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{pmatrix} \text{ otherwise,}$$

and if $n_3 > n_2$, then

$$E^{(n_2)} := \begin{pmatrix} E_{11} \\ E_{12} \end{pmatrix}.$$

Then the Smith normal form (see [24], [17] or Section 2 below) guarantees the existences of unimodular matrices $U^{(n_l)}$ and $V^{(n_l)}$ such that

$$U^{(n_l)} E^{(n_l)} V^{(n_l)} = \begin{pmatrix} D^{(n_l)} & 0 \\ 0 & 0 \end{pmatrix},$$

where $D^{(n_l)} := \text{diag}(d_1^{(l)}, \dots, d_{s_l}^{(l)})$ with all the diagonal elements $d_1^{(l)}, \dots, d_{s_l}^{(l)}$ being positive integers such that $d_1^{(l)} | \dots | d_{s_l}^{(l)}$. Throughout, pick $\alpha \in \mathbb{F}_q^*$ to be a fixed primitive element of \mathbb{F}_q . For any $\beta \in \mathbb{F}_q^*$, there exists exactly an integer $r \in [1, q-1]$ such that $\beta = \alpha^r$. Such an integer r is called *index* of β with respect to the primitive element α , and is denoted by $\text{ind}_\alpha(\beta) := r$.

For $l = 1, \dots, 4$, we let $SE^{(n_l)}$ denote the system (1.4) which deletes the items containing the variable x_{n_l+1} . Clearly the exponent matrix of $SE^{(n_l)}$ is $E^{(n_l)}$. For example, if $l = 1$ and $n_1 < n_3$, then $SE^{(n_1)}$ becomes the following system of equations over \mathbb{F}_q :

$$\begin{cases} \sum_{i=1}^{r_1} a_{1i} x_1^{e_{i1}^{(1)}} \dots x_{n_1}^{e_{i,n_1}^{(1)}} - b_1 = 0, \\ b_2 = 0, \end{cases}$$

where a_{1i} , b_1 , b_2 and $e_{ij}^{(1)}$ are given in (1.4) for all integers i and j with $1 \leq i \leq r_1$ and $1 \leq j \leq n_1$. Let $x_1^{e_{i1}^{(1)}} \dots x_{n_1}^{e_{i,n_1}^{(1)}} = u_{1i}$ for any integer i with $1 \leq i \leq r_1$. If $b_2 \neq 0$, then it is obvious that $SE^{(n_1)}$ has no solution. If $b_2 = 0$, then $SE^{(n_1)}$ becomes

$$\sum_{i=1}^{r_1} a_{1i} u_{1i} - b_1 = 0. \quad (\overline{SE}^{(n_1)})$$

Lemma 2.4 tells us the formula for the number of the solution $(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}$ of $(\overline{SE}^{(n_1)})$. Now we use H_1 to denote the number of the solutions $(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}$ of $(\overline{SE}^{(n_1)})$ under the following extra conditions:

$$\begin{cases} \gcd(q-1, d_i^{(1)}) | h'_i & \text{for } i = 1, \dots, s_1, \\ (q-1) | h'_i & \text{for } i = s_1 + 1, \dots, r_1, \end{cases} \quad (\widetilde{SE}^{(n_1)})$$

where $(h'_1, \dots, h'_{r_1})^T = U^{(n_1)}(\text{ind}_\alpha(u_{11}), \dots, \text{ind}_\alpha(u_{1r_1}))^T$. Similarly, for $l = 2, 3, 4$, we can get the system of equations $SE^{(n_l)}$ and use H_l to denote the number of the solutions of $(\overline{SE}^{(n_l)})$ under the extra conditions $(\widetilde{SE}^{(n_l)})$. One knows that H_l (see [15]) is independent of the choice of the primitive element α . In what follows, we let N stand for the number of rational points on the algebraic variety defined by (1.4).

Now let

$$N_0 := q^{\max\{n_2, n_4\} - n_1} (q^{n_1} - (q-1)^{n_1}) \text{ if } b_1 = b_2 = 0, \text{ and } 0 \text{ otherwise,}$$

$$N_1 := \begin{cases} q^{\max\{n_2, n_4\} - \min\{n_2, n_3\}} (q^{\min\{n_2, n_3\} - n_1} - (q-1)^{\min\{n_2, n_3\} - n_1}) L_1, & \text{if } n_1 < n_3 \text{ and } b_2 = 0, \\ q^{\max\{n_2, n_4\} - \min\{n_2, n_4\}} (q^{\min\{n_2, n_4\} - n_1} - (q-1)^{\min\{n_2, n_4\} - n_1}) L_1, & \text{if } n_1 = n_3 \text{ and } b_2 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$N_2 := q^{\max\{n_2, n_4\} - \min\{n_2, n_4\}} (q^{\min\{n_2, n_4\} - n_3} - (q-1)^{\min\{n_2, n_4\} - n_3}) L_3$$

if $n_3 < n_2$, and 0 otherwise,

$$N_3 := \begin{cases} (q^{n_4 - n_2} - (q-1)^{n_4 - n_2}) L_2, & \text{if } n_1 \leq n_3 < n_2 < n_4, \\ (q^{n_4 - n_3} - (q-1)^{n_4 - n_3}) L_3, & \text{if } n_3 \geq n_2, \\ 0, & \text{otherwise,} \end{cases}$$

$N_4 := L_4$ if $n_4 \geq n_2$, and L_2 otherwise, $N_5 := (q^{n_2 - n_4} - (q-1)^{n_2 - n_4}) L_4$ if $n_2 > n_4$, and 0 otherwise, $N_6 := q^{n_4 - n_3} (q^{n_3 - n_2} - (q-1)^{n_3 - n_2}) L_2$ if $n_3 > n_2$, and $b_2 = 0$, and 0 otherwise, where for $1 \leq i \leq 4$,

$$L_i := H_i(q-1)^{n_i - s_i} \prod_{j=1}^{s_i} \gcd(q-1, d_j^{(i)}).$$

We can now state the main result of this paper which answers Problem 1.2 completely.

Theorem 1.3. *Let N denote the number of rational points on the algebraic variety (1.4). Then*

$$N = \begin{cases} \sum_{i=0}^4 N_i, & \text{if } n_1 < n_3 < n_2 < n_4, \\ \sum_{i=0}^4 N_i, & \text{if } n_2 = n_3, \\ \sum_{i=0}^4 N_i, & \text{if } n_1 = n_3, n_2 < n_4, \\ \sum_{i=0}^4 N_i, & \text{if } n_1 < n_3, n_2 = n_4, \\ \sum_{i=0}^4 N_i, & \text{if } n_1 = n_3, n_2 = n_4, \\ \sum_{i=0}^5 N_i, & \text{if } n_1 < n_3, n_2 > n_4, \\ \sum_{i=0}^5 N_i, & \text{if } n_1 = n_3, n_2 > n_4, \\ \sum_{i=0}^6 N_i, & \text{if } n_3 > n_2. \end{cases}$$

with $N_i (0 \leq i \leq 6)$ being defined as above.

We organize this paper as follows. In Section 2, we present some useful lemmas which will be needed later. In fact, we will first recall some basic facts on the Smith normal form of an integer matrix. Then we can use them to give a formula for the number of the system of linear congruences with the same modulo. Consequently, in Section 3, we first show a key lemma and then use it to prove Theorem 1.3. In the final section, we provide an example to demonstrate the validity of Theorem 1.3.

Throughout this paper, $\gcd(a, m)$ will denote the greatest common divisor of any positive integers a and m .

2 Preliminary lemmas

In this section, we present some useful lemmas that are needed in Section 3. We first recall two well-known definitions.

Definition 2.1. [17] Let M be a square integer matrix. If the determinant of M is ± 1 , then M is called an *unimodular matrix*.

Definition 2.2. [17] For given any positive integers m and n , let P and Q be two $m \times n$ integer matrices. Suppose that there are two modular matrices U of order m and V of order n such that $P = UQV$. Then we say that P and Q are *equivalent* and we write $P \sim Q$.

Clearly equivalence has the three properties of being reflexive, symmetric and transitive.

Lemma 2.1. [17] [24] Let P be a nonzero $m \times n$ integer matrix. Then P is equivalent to a block matrix of the following form

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.1)$$

where $D = \text{diag}(d_1, \dots, d_r)$ with all the diagonal elements d_i being positive integers and satisfying that $d_i | d_{i+1}$ ($1 \leq i < r$). In other words, there are unimodular matrices U of order m and V of order n such that

$$UPV = \begin{pmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{pmatrix}.$$

We call the diagonal matrix in (2.1) the *Smith normal form* of the matrix P . Usually, one writes the Smith normal form of P as $\text{SNF}(P)$. The elements d_i are unique up to multiplication by a unit and are called the *elementary divisors*, *invariants*, or *invariant factors*.

For any system of linear congruences

$$\begin{cases} \sum_{j=1}^n h_{1j} y_j & \equiv b_1 \pmod{m}, \\ \dots\dots\dots \\ \sum_{j=1}^n h_{sj} y_j & \equiv b_s \pmod{m}, \end{cases} \quad (2.2)$$

let $Y = (y_1, \dots, y_n)^T$ be the column of indeterminates y_1, \dots, y_n , $B = (b_1, \dots, b_s)^T$ and $H = (h_{ij})$ be the matrix of its coefficient. Then one can write (2.2) as

$$HY \equiv B \pmod{m}. \quad (2.3)$$

By Lemma 2.1, there are unimodular matrices U of order s and V of order n satisfying that

$$UHV = \text{SNF}(H) = \begin{pmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, we present the following two known lemmas.

Lemma 2.2. [16] *Let $B' = (b'_1, \dots, b'_s)^T = UB$. Then the system (2.3) of linear congruences is solvable if and only if $\gcd(m, d_i) | b'_i$ for all integers i with $1 \leq i \leq r$ and $m | b'_i$ for all integers i with $r+1 \leq i \leq s$. Further, the number of solutions of (2.3) is equal to $m^{n-r} \prod_{i=1}^r \gcd(m, d_i)$.*

Lemma 2.3. [27] *Let $c_1, \dots, c_k \in \mathbb{F}_q^*$ and $c \in \mathbb{F}_q$, and let $N(c)$ denote the number of rational points $(x_1, \dots, x_k) \in (\mathbb{F}_q^*)^k$ on the hypersurface $c_1x_1 + \dots + c_kx_k = c$. Then*

$$N(c) = \begin{cases} \frac{(q-1)^k + (-1)^k(q-1)}{q}, & \text{if } c = 0, \\ \frac{(q-1)^k - (-1)^k}{q}, & \text{otherwise.} \end{cases}$$

Lemma 2.4. *Let $c_{ij} \in \mathbb{F}_q^*$ for all integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq k$ and $c_1, \dots, c_m \in \mathbb{F}_q$. Let $N(c_1, \dots, c_m)$ denote the number of rational points $(x_{11}, \dots, x_{1k}, \dots, x_{m1}, \dots, x_{mk}) \in (\mathbb{F}_q^*)^{mk}$ on the following variety*

$$\begin{cases} c_{11}x_{11} + \dots + c_{1k}x_{1k} = c_1, \\ \dots \\ c_{m1}x_{m1} + \dots + c_{mk}x_{mk} = c_m. \end{cases} \quad (2.4)$$

Then

$$N(c_1, \dots, c_m) = \frac{(q-1)^r}{q^m} ((q-1)^{k-1} + (-1)^k)^r ((q-1)^k - (-1)^k)^{m-r},$$

where $r := \#\{1 \leq i \leq m | c_i = 0\}$.

Proof. For $1 \leq i \leq m$, let $N(c_i)$ denote the number of rational points $(x_{i1}, \dots, x_{ik}) \in (\mathbb{F}_q^*)^k$ on the hypersurface $c_{i1}x_{i1} + \dots + c_{ik}x_{ik} = c_i$. Since for any rational points $(x_{11}, \dots, x_{1k}, \dots, x_{m1}, \dots, x_{mk}) \in (\mathbb{F}_q^*)^{mk}$ on the variety (2.4), the different part of coordinates $(x_{i1}, \dots, x_{ik}) \in (\mathbb{F}_q^*)^k (1 \leq i \leq m)$ are independent, one has

$$N(c_1, \dots, c_m) = \prod_{i=1}^m N(c_i).$$

So Lemma 2.3 applied to $N(c_i)$ gives us the required result. Hence Lemma 2.4 is proved. \square

3 Proof of Theorem 1.3

In this section, we show Theorem 1.3. First, we provide some notation and two lemmas. For any given $(u_{11}, \dots, u_{1r_2}, \dots, u_{21}, \dots, u_{2r_4}) \in \mathbb{F}_q^{r_2+r_4}$, we use

$$N(x^{E_i^{(1)}} = u_{1i} (i = 1, 2, \dots, r_2), x^{E_j^{(2)}} = u_{2j} (j = 1, 2, \dots, r_4)).$$

to denote the number of rational points $(x_1, \dots, x_{\max\{n_2, n_4\}}) \in \mathbb{F}_q^{\max\{n_2, n_4\}}$ of the following algebraic variety over \mathbb{F}_q :

$$\begin{cases} x^{E_1^{(1)}} = u_{11}, \\ \dots\dots\dots \\ x^{E_{r_2}^{(1)}} = u_{1r_2}, \\ x^{E_1^{(2)}} = u_{21}, \\ \dots\dots\dots \\ x^{E_{r_4}^{(2)}} = u_{2r_4}. \end{cases} \quad (3.1)$$

It follows that

$$N = \sum_{\substack{(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in \mathbb{F}_q^{r_2+r_4}, \\ a_{11}u_{11} + \dots + a_{1r_2}u_{1r_2} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_4}u_{2r_4} = b_2.}} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i} (i = 1, 2, \dots, r_2), \\ x^{E_j^{(2)}} = u_{2j} (j = 1, 2, \dots, r_4). \end{cases}\right).$$

Define

$$T := \{(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in \mathbb{F}_q^{r_2+r_4} : \begin{matrix} a_{11}u_{11} + \dots + a_{1r_2}u_{1r_2} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_4}u_{2r_4} = b_2. \end{matrix}\},$$

with b_1, b_2, a_{1i} ($1 \leq i \leq r_2$) and a_{2j} ($1 \leq j \leq r_4$) being given as in (1.4). Then

$$N = \sum_{(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i} (i = 1, 2, \dots, r_2), \\ x^{E_j^{(2)}} = u_{2j} (j = 1, 2, \dots, r_4). \end{cases}\right). \quad (3.2)$$

Let $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T$. We consider the algebraic variety of (3.1). For integer i, j with $1 \leq i < r_2$ and $1 \leq j < r_4$, one can easily deduce that

$$u_{1,i+1} = 0 \text{ if } u_{1i} = 0, \text{ and } u_{2,j+1} = 0 \text{ if } u_{2j} = 0. \quad (3.3)$$

Define

$$V_1 := \{(u_{11}, \dots, u_{1r_1}, 0, \dots, 0) \in \mathbb{F}_q^{r_2+r_4} : u_{1i} \in \mathbb{F}_q^*, 1 \leq i \leq r_1\},$$

$$V_2 := \{(u_{11}, \dots, u_{1r_1}, \underbrace{0, \dots, 0}_{r_2-r_1}, u_{21}, \dots, u_{2r_3}, 0, \dots, 0) \in \mathbb{F}_q^{r_2+r_4} : u_{1i}, u_{2j} \in \mathbb{F}_q^*, 1 \leq i \leq r_1, 1 \leq j \leq r_3\},$$

$$V_3 := \{(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_3}, 0, \dots, 0) \in \mathbb{F}_q^{r_2+r_4} : u_{1i}, u_{2j} \in \mathbb{F}_q^*, 1 \leq i \leq r_2, 1 \leq j \leq r_3\},$$

$$V_4 := \{(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in \mathbb{F}_q^{r_2+r_4} : u_{1i}, u_{2j} \in \mathbb{F}_q^*, 1 \leq i \leq r_2, 1 \leq j \leq r_4\},$$

$$V_5 := \{(u_{11}, \dots, u_{1r_1}, 0, \dots, 0, u_{21}, \dots, u_{2r_4}) \in \mathbb{F}_q^{r_2+r_4} : u_{1i}, u_{2j} \in \mathbb{F}_q^*, 1 \leq i \leq r_1, 1 \leq j \leq r_4\},$$

and

$$V_6 := \{(u_{11}, \dots, u_{1r_2}, 0, \dots, 0) \in \mathbb{F}_q^{r_2+r_4} : u_{1i} \in \mathbb{F}_q^*, 1 \leq i \leq r_2\}.$$

Let $T(V_0)$ consist of zero vector of dimension $r_2 + r_4$ and $T(V_i)$ ($0 \leq i \leq 6$) denote the set of the vectors v such that $v \in T$ and $v \in V_i$. For any integer i with $0 \leq i \leq 6$, define

$$M_i := \sum_{(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T(V_i)} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i} (i = 1, 2, \dots, r_2), \\ x^{E_j^{(2)}} = u_{2j} (j = 1, 2, \dots, r_4). \end{cases}\right). \quad (3.4)$$

Then we have the following key lemma.

Lemma 3.1. *One has that $M_i = N_i$ for all $0 \leq i \leq 6$.*

Proof. First we show that $M_0 = N_0$. For any given integer i with $i = 1, 2$, if there exists $b_i \neq 0$, then the algebraic variety

$$\begin{cases} a_{11}u_{11} + \dots + a_{1r_2}u_{1r_2} - b_1 = 0, \\ a_{21}u_{21} + \dots + a_{2r_4}u_{2r_4} - b_2 = 0, \end{cases}$$

has no zero solutions $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in \mathbb{F}_q^{r_2+r_4}$. So $T(V_0)$ is empty. It follows that $M_0 = 0 = N_0$. If $b_1 = b_2 = 0$, then $T(V_0)$ consists of zero vector of dimension $r_2 + r_4$. Then

$$\begin{aligned} M_0 &= N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i} (i = 1, \dots, r_2) \\ x^{E_j^{(2)}} = u_{2j} (j = 1, \dots, r_4) \end{cases}\right) \\ &= N(x^{E_1^{(1)}} = 0) \quad (\text{since } n_1 \leq n_3) \\ &= N(x_1^{e_{11}^{(1)}} \dots x_{n_1}^{e_{1n_1}^{(1)}} = 0) \\ &= q^{\max\{n_2, n_4\} - n_1} \sum_{j=1}^{n_1} \binom{n_1}{j} (q-1)^{n_1-j} \\ &= q^{\max\{n_2, n_4\} - n_1} (q^{n_1} - (q-1)^{n_1}) = N_0. \end{aligned}$$

This completes the proof of the first part of the Lemma 3.1.

We can now turn our attention to prove that $M_1 = N_1$. First we consider the case $n_1 < n_3 < n_2$. Since $n_1 < n_3$, it follows from (3.4) and the definition of $T(V_1)$ that

$$M_1 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1 \text{ and } 0 = b_2.}} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, \quad 1 \leq i \leq r_1, \\ x^{E_j^{(1)}} = 0, \quad r_1 + 1 \leq j \leq r_2, \\ x^{E_k^{(2)}} = 0, \quad 1 \leq k \leq r_4. \end{cases}\right). \quad (3.5)$$

From (3.5), we deduce that if $b_2 \neq 0$, then $M_1 = 0 = N_1$. That is,

$$M_1 = N_1 \text{ if } n_1 < n_3 < n_2 \text{ and } b_2 \neq 0.$$

Now we let $b_2 = 0$. Notice that $n_3 < n_2$, then the definitions of $x^{E_j^{(1)}}$ and $x^{E_k^{(2)}}$ tell us that the fact that $x^{E_j^{(1)}} = 0$ and $x^{E_k^{(2)}} = 0$ with $r_1 + 1 \leq j \leq r_2$, $1 \leq k \leq r_4$ is reduced to saying that $x^{E_1^{(2)}} = 0$. It then follows from (3.5) that

$$M_1 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1.}} N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x^{E_1^{(2)}} = 0). \quad (3.6)$$

It is easy to see that $x^{E_1^{(2)}} = 0$ is equivalent to $x_1 \dots x_{n_3} = 0$. Since $u_{1r_1} \neq 0$ and $u_{1r_1} = x_1^{e_{r_1,1}^{(1)}} \dots x_{n_1}^{e_{r_1,n_1}^{(1)}}$, one has $x_1 \dots x_{n_1} \neq 0$. So $x^{E_1^{(2)}} = 0$ is equivalent to $x_{n_1+1} \dots x_{n_3} = 0$. Then by (3.6), one gets that

$$M_1 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1.}} N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x_{n_1+1} \dots x_{n_3} = 0). \quad (3.7)$$

For any given $(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}$ with $\sum_{i=1}^{r_1} a_{1i}u_{1i} = b_1$, one has

$$\begin{aligned} & N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x_{n_1+1} \dots x_{n_3} = 0) \\ &= \#\{(x_1, \dots, x_{\max\{n_2, n_4\}}) \in (\mathbb{F}_q)^{\max\{n_2, n_4\}} : x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x_{n_1+1} \dots x_{n_3} = 0\}. \end{aligned}$$

Since each of the components $x_{n_3+1}, \dots, x_{\max\{n_2, n_4\}}$ can run over the whole finite field \mathbb{F}_q independently, it then follows that

$$\begin{aligned} & N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x_{n_1+1} \dots x_{n_3} = 0) \\ &= q^{\max\{n_2, n_4\} - n_3} \times \\ & \#\{(x_1, \dots, x_{n_3}) \in (\mathbb{F}_q)^{n_3} : x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x_{n_1+1} \dots x_{n_3} = 0\}. \end{aligned} \quad (3.8)$$

Notice that the choice of $(x_1, \dots, x_{n_1}) \in (\mathbb{F}_q^*)^{n_1}$ satisfying that $x^{E_i^{(1)}} = u_{1i}$ ($i = 1, \dots, r_1$) is independent of the choice of $(x_{n_1+1}, \dots, x_{n_3}) \in (\mathbb{F}_q)^{n_3-n_1}$ satisfying that $x_{n_1+1} \dots x_{n_3} = 0$. We then derive that

$$\begin{aligned} & \#\{(x_1, \dots, x_{n_3}) \in (\mathbb{F}_q)^{n_3} : x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x_{n_1+1} \dots x_{n_3} = 0\} \\ &= \#\{(x_1, \dots, x_{n_1}) \in (\mathbb{F}_q^*)^{n_1} : x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1\} \times \\ & \#\{(x_{n_1+1}, \dots, x_{n_3}) \in (\mathbb{F}_q)^{n_3-n_1} : x_{n_1+1} \dots x_{n_3} = 0\}. \end{aligned} \quad (3.9)$$

On the other hand, we can easily compute that

$$\begin{aligned} & \#\{(x_{n_1+1}, \dots, x_{n_3}) \in (\mathbb{F}_q)^{n_3-n_1} : x_{n_1+1} \dots x_{n_3} = 0\} \\ &= \sum_{i=1}^{n_3-n_1} \binom{n_3-n_1}{i} (q-1)^{n_3-n_1-i} = q^{n_3-n_1} - (q-1)^{n_3-n_1}. \end{aligned} \quad (3.10)$$

So by (3.8) to (3.10), one obtains that

$$\begin{aligned}
& N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1 \text{ and } x_{n_1+1} \dots x_{n_3} = 0) \\
&= q^{\max\{n_2, n_4\} - n_3} (q^{n_3 - n_1} - (q - 1)^{n_3 - n_1}) \\
&\times \#\{(x_1, \dots, x_{n_1}) \in (\mathbb{F}_q^*)^{n_1} : x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1\} \\
&= q^{\max\{n_2, n_4\} - n_3} (q^{n_3 - n_1} - (q - 1)^{n_3 - n_1}) N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1). \tag{3.11}
\end{aligned}$$

Then by (3.7) together with (3.11), we have

$$\begin{aligned}
M_1 &= q^{\max\{n_2, n_4\} - n_3} (q^{n_3 - n_1} - (q - 1)^{n_3 - n_1}) \times \\
&\sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1}} N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1). \tag{3.12}
\end{aligned}$$

Now we treat with the sum

$$\sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1}} N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1). \tag{3.13}$$

For any given $(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}$ with $\sum_{i=1}^{r_1} a_{1i}u_{1i} = b_1$, one has that

$$N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1)$$

equals the number of the solutions $(x_1, \dots, x_{n_1}) \in (\mathbb{F}_q^*)^{n_1}$ of the following system of equations:

$$\begin{cases} x_1^{e_{11}^{(1)}} \dots x_{n_1}^{e_{1n_1}^{(1)}} = u_{11}, \\ \dots\dots\dots \\ x_1^{e_{r_1,1}^{(1)}} \dots x_{n_1}^{e_{r_1,n_1}^{(1)}} = u_{1r_1}. \end{cases} \tag{3.14}$$

Since $u_{1i} \neq 0$ ($1 \leq i \leq r_1$), we can deduce that the number of the solutions $(x_1, \dots, x_{n_1}) \in (\mathbb{F}_q^*)^{n_1}$ of (3.14) is equal to the number of nonnegative integral solutions $(\text{ind}_\alpha(x_1), \dots, \text{ind}_\alpha(x_{n_1})) \in \mathbb{N}^{n_1}$ of the following system of congruences

$$\begin{cases} \sum_{i=1}^{n_1} e_{1i}^{(1)} \text{ind}_\alpha(x_i) \equiv \text{ind}_\alpha(u_{11}) \pmod{q-1}, \\ \dots\dots\dots \\ \sum_{i=1}^{n_1} e_{r_1,i}^{(1)} \text{ind}_\alpha(x_i) \equiv \text{ind}_\alpha(u_{1r_1}) \pmod{q-1}. \end{cases} \tag{3.15}$$

But Lemma 2.2 tells us that (3.15) has solutions $(\text{ind}_\alpha(x_1), \dots, \text{ind}_\alpha(x_{n_1})) \in \mathbb{N}^{n_1}$ if and only if the extra conditions $(\widetilde{SE}^{(n_1)})$ hold. Further, Lemma 2.2 gives us

the number of solutions $(\text{ind}_\alpha(x_1), \dots, \text{ind}_\alpha(x_{n_1})) \in \mathbb{N}^{n_1}$ of (3.15) which is equal to

$$(q-1)^{n_1-s_1} \prod_{i=1}^{s_1} \gcd(q-1, d_i^{(1)}).$$

Hence

$$N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1) = (q-1)^{n_1-s_1} \prod_{i=1}^{s_1} \gcd(q-1, d_i^{(1)}). \quad (3.16)$$

Since

$$H_1 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1 \\ \text{and } (\widetilde{SE}^{(n_1)}) \text{ holds.}}} 1,$$

it then follows from (3.16) that

$$\begin{aligned} & \sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1.}} N(x^{E_i^{(1)}} = u_{1i}, 1 \leq i \leq r_1) \\ &= \sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1 \\ \text{and } (\widetilde{SE}^{(n_1)}) \text{ holds.}}} (q-1)^{n_1-s_1} \prod_{i=1}^{s_1} \gcd(q-1, d_i^{(1)}) \\ &= (q-1)^{n_1-s_1} \prod_{i=1}^{s_1} \gcd(q-1, d_i^{(1)}) \sum_{\substack{(u_{11}, \dots, u_{1r_1}) \in (\mathbb{F}_q^*)^{r_1}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1 \\ \text{and } (\widetilde{SE}^{(n_1)}) \text{ holds.}}} 1 \\ &= H_1 (q-1)^{n_1-s_1} \prod_{i=1}^{s_1} \gcd(q-1, d_i^{(1)}). \end{aligned} \quad (3.17)$$

Thus by (3.6), (3.12) and (3.17), we obtain that $M_1 = N_1$ if $n_1 < n_3 < n_2$ and $b_2 = 0$.

By the same argument, one can deduce that either $n_3 > n_2$ or $n_1 = n_3$ we have $M_1 = N_1$. This ends the second part of the Lemma 3.1.

Now we treat the third part of the Lemma 3.1. From (3.3) and the definition of $T(V_2)$, we know that for any vector $v \in T(V_2)$ if and only if $n_3 < n_2$. Thus by the definition of N_2 and (3.4), we get that $M_2 = N_2 = 0$ if $n_3 \geq n_2$.

Let $n_3 < n_2$. If $n_1 < n_3 < n_2 < n_4$, it then follows from (3.4) and the definition of $T(V_2)$ that

$$M_2 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}, u_{21}, \dots, u_{2r_3}) \in (\mathbb{F}_q^*)^{r_1+r_3}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_3}u_{2r_3} = b_2.}} N \left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(1)}} = 0, & r_1 + 1 \leq j \leq r_2, \\ x^{E_i^{(2)}} = u_{2i}, & 1 \leq i \leq r_3, \\ x^{E_k^{(2)}} = 0, & r_3 + 1 \leq k \leq r_4. \end{cases} \right). \quad (3.18)$$

The definitions of $x^{E_j^{(1)}}$ and $x^{E_k^{(2)}}$ tell us that the fact that $x^{E_j^{(1)}} = 0$ and $x^{E_k^{(2)}} = 0$ with $r_1 + 1 \leq j \leq r_2$, $r_3 + 1 \leq k \leq r_4$ is reduced to saying that $x^{E_{r_1+1}^{(1)}} = 0$. It then follows from (3.18) that

$$M_2 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}, u_{21}, \dots, u_{2r_3}) \in (\mathbb{F}_q^*)^{r_1+r_3}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_3}u_{2r_3} = b_2.}} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x^{E_{r_1+1}^{(1)}} = 0. \end{cases}\right). \quad (3.19)$$

It is easy to see that $x^{E_{r_1+1}^{(1)}} = 0$ is equivalent to $x_1 \dots x_{n_2} = 0$. Since $n_3 < n_2$, $u_{2r_3} \neq 0$ and $u_{2r_3} = x_1^{e_{r_3,1}^{(2)}} \dots x_{n_3}^{e_{r_3,n_3}^{(2)}}$, one has $x_1 \dots x_{n_3} \neq 0$. So $x^{E_{r_1+1}^{(1)}} = 0$ is equivalent to $x_{n_3+1} \dots x_{n_2} = 0$. Then by (3.19), one gets that

$$M_2 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}, u_{21}, \dots, u_{2r_3}) \in (\mathbb{F}_q^*)^{r_1+r_3}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_3}u_{2r_3} = b_2.}} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x_{n_3+1} \dots x_{n_2} = 0. \end{cases}\right). \quad (3.20)$$

For any given $(u_{11}, \dots, u_{1r_1}, u_{21}, \dots, u_{2r_3}) \in (\mathbb{F}_q^*)^{r_1+r_3}$ with $\sum_{i=1}^{r_1} a_{1i}u_{1i} = b_1$ and $\sum_{j=1}^{r_3} a_{2j}u_{2j} = b_2$, one has

$$N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x_{n_3+1} \dots x_{n_2} = 0. \end{cases}\right) = \#\{(x_1, \dots, x_{n_4}) \in (\mathbb{F}_q)^{n_4} : \begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x_{n_3+1} \dots x_{n_2} = 0. \end{cases}\}.$$

Since each of the components $x_{n_2+1}, \dots, x_{n_4}$ can run over the whole finite field \mathbb{F}_q independently, it then follows that

$$\begin{aligned} & N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x_{n_3+1} \dots x_{n_2} = 0. \end{cases}\right) \\ &= q^{n_4 - n_2} \times \#\{(x_1, \dots, x_{n_2}) \in (\mathbb{F}_q)^{n_2} : \begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x_{n_3+1} \dots x_{n_2} = 0. \end{cases}\}. \end{aligned} \quad (3.21)$$

Notice that the choice of $(x_1, \dots, x_{n_3}) \in (\mathbb{F}_q^*)^{n_3}$ satisfying that $x^{E_i^{(1)}} = u_{1i}$ ($i = 1, \dots, r_1$) and $x^{E_j^{(2)}} = u_{2j}$ ($j = 1, \dots, r_3$) is independent of the choice of $(x_{n_3+1}, \dots, x_{n_2}) \in (\mathbb{F}_q)^{n_2 - n_3}$ satisfying that $x_{n_3+1} \dots x_{n_2} = 0$. We then derive

that

$$\begin{aligned}
& \#\{(x_1, \dots, x_{n_2}) \in (\mathbb{F}_q)^{n_2} : \begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x_{n_3+1} \dots x_{n_2} = 0. \end{cases} \} \\
&= \#\{(x_1, \dots, x_{n_3}) \in (\mathbb{F}_q)^{n_3} : \begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3. \end{cases} \} \times \\
&\#\{(x_{n_3+1}, \dots, x_{n_2}) \in (\mathbb{F}_q)^{n_2-n_3} : x_{n_3+1} \dots x_{n_2} = 0\}. \tag{3.22}
\end{aligned}$$

On the other hand, we can easily compute that

$$\#\{(x_{n_3+1}, \dots, x_{n_2}) \in (\mathbb{F}_q)^{n_2-n_3} : x_{n_3+1} \dots x_{n_2} = 0\} = q^{n_2-n_3} - (q-1)^{n_2-n_3}. \tag{3.23}$$

So by (3.21) to (3.23), one obtains that

$$\begin{aligned}
& N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3, \\ x_{n_3+1} \dots x_{n_2} = 0. \end{cases}\right) \\
&= q^{n_4-n_2}(q^{n_2-n_3} - (q-1)^{n_2-n_3}) \times \\
&\#\{(x_1, \dots, x_{n_3}) \in (\mathbb{F}_q)^{n_3} : \begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3. \end{cases} \} \\
&= q^{n_4-n_2}(q^{n_2-n_3} - (q-1)^{n_2-n_3}) N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3. \end{cases}\right). \tag{3.24}
\end{aligned}$$

Then by (3.20) together with (3.24), we have

$$\begin{aligned}
M_2 &= q^{n_4-n_2}(q^{n_2-n_3} - (q-1)^{n_2-n_3}) \times \\
&\sum_{\substack{(u_{11}, \dots, u_{1r_1}, u_{21}, \dots, u_{2r_3}) \in (\mathbb{F}_q^*)^{r_1+r_3}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_3}u_{2r_3} = b_2.}} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3. \end{cases}\right). \tag{3.25}
\end{aligned}$$

Now the argument for (3.13) applied to the sum

$$\sum_{\substack{(u_{11}, \dots, u_{1r_1}, u_{21}, \dots, u_{2r_3}) \in (\mathbb{F}_q^*)^{r_1+r_3}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_3}u_{2r_3} = b_2.}} N\left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(2)}} = u_{2j}, & 1 \leq j \leq r_3. \end{cases}\right)$$

gives us that

$$M_2 = q^{n_4 - n_2} (q^{n_2 - n_3} - (q - 1)^{n_2 - n_3}) H_3 (q - 1)^{n_3 - s_3} \prod_{i=1}^{s_3} \gcd(q - 1, d_i^{(3)}) = N_2.$$

This concludes the proof of the third part of the Lemma 3.1.

Next we prove the fourth part of Lemma 3.1. The definition of $T(V_3)$ together with (3.3) give us that for any vector $v \in T(V_3)$ if and only if $n_1 \leq n_3 < n_2 < n_4$ or $n_2 \leq n_3$. Using the same argument of the third part of Lemma 3.1, one can get that either $n_1 \leq n_3 < n_2 < n_4$ or $n_2 \leq n_3$, $M_3 = N_3$. For the other case, one has $M_3 = N_3 = 0$.

Now we deal with the fifth part of Lemma 3.1. It follows (3.4) and the definition of $T(V_4)$ that

$$M_4 = \sum_{\substack{(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in (\mathbb{F}_q^*)^{r_2 + r_4}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_3}u_{2r_3} = b_2.}} N \left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_2, \\ x^{E_j^{(2)}} = u_{2i}, & 1 \leq i \leq r_4. \end{cases} \right). \quad (3.26)$$

If $n_2 \leq n_4$, the argument of (3.13) applied to (3.26) tells us that

$$M_4 = H_4 (q - 1)^{n_4 - s_4} \prod_{j=1}^{s_4} \gcd(q - 1, d_j^{(4)}) = N_4.$$

Similarly, if $n_2 > n_4$, we can get that

$$M_4 = H_2 (q - 1)^{n_2 - s_2} \prod_{j=1}^{s_2} \gcd(q - 1, d_j^{(2)}) = N_4.$$

Next we prove the sixth part of Lemma 3.1. The definition of $T(V_5)$ and (3.3) give us that the vector $v \in T(V_5)$ if and only if $n_2 > n_4$. If $n_2 \leq n_4$, by the definition of M_5 , one can easily deduce that $M_5 = N_5 = 0$. Now we let $n_2 > n_4$. It follows from (3.4) and the definition of $T(V_5)$ that

$$M_5 = \sum_{\substack{(u_{11}, \dots, u_{1r_1}, u_{21}, \dots, u_{2r_4}) \in (\mathbb{F}_q^*)^{r_1 + r_4}, \\ a_{11}u_{11} + \dots + a_{1r_1}u_{1r_1} = b_1, \\ a_{21}u_{21} + \dots + a_{2r_4}u_{2r_4} = b_2.}} N \left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_1, \\ x^{E_j^{(1)}} = 0, & r_1 + 1 \leq j \leq r_2, \\ x^{E_k^{(2)}} = u_{2k}, & 1 \leq k \leq r_4. \end{cases} \right). \quad (3.27)$$

Using the same argument of (3.19), we obtain that

$$M_5 = (q^{n_2 - n_4} - (q - 1)^{n_2 - n_4}) H_4 (q - 1)^{n_4 - s_4} \prod_{j=1}^{s_4} \gcd(q - 1, d_j^{(4)}) = N_5.$$

In the rest, we prove $M_6 = N_6$. From the definition of $T(V_6)$ and (3.3), we know that the vector $v \in T(V_6)$ if and only if $n_3 > n_2$. Thus one has

$M_6 = N_6 = 0$, if $n_3 \leq n_2$. Now we let $n_3 > n_2$. Then (3.4) and the definition of $T(V_6)$ tell us that

$$M_6 = \sum_{\substack{(u_{11}, \dots, u_{1r_2}) \in (\mathbb{F}_q^*)^{r_2}, \\ a_{11}u_{11} + \dots + a_{1r_2}u_{1r_2} = b_1 \text{ and } b_2 = 0.}} N \left(\begin{cases} x^{E_i^{(1)}} = u_{1i}, & 1 \leq i \leq r_2, \\ x^{E_j^{(2)}} = 0, & 1 \leq j \leq r_4. \end{cases} \right). \quad (3.28)$$

Using the same argument of (3.5), we get that

$$M_6 = \begin{cases} q^{n_4 - n_3} (q^{n_3 - n_2} - (q - 1)^{n_3 - n_2}) H_2(q - 1)^{n_2 - s_2} \prod_{j=1}^{s_2} \gcd(q - 1, d_j^{(2)}), & \text{if } n_3 > n_2 \text{ and } b_2 = 0, \\ 0, & \text{otherwise,} \end{cases}$$

as desired. This completes the proof of the Lemma 3.1. \square

We can now use Lemma 3.1 to prove Theorem 1.3 as the conclusion of this section.

Proof of Theorem 1.3. We divide the proof into the following eight cases.

Case 1. $n_1 < n_3 < n_2 < n_4$. From (3.3) we can obtain that if the vectors $v \in T$ have nonzero components of the vectors $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T$, then we must have $v \in v_1$, or $v \in v_2$, or $v \in v_3$, or $v \in v_4$. Then we have $T = \bigcup_{i=0}^4 T(V_i)$. It follows from (3.2) and (3.4) that $N = \sum_{i=0}^4 M_i$. So by the

Lemma 3.1, we have $N = \sum_{i=0}^4 N_i$ as desired.

Case 2. $n_2 = n_3$. Since (3.3) tells us that if the vectors $v \in T$ have nonzero components of the vectors $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T$, then we must have $v \in v_1$, or $v \in v_3$, or $v \in v_4$. Then $T = \bigcup_{\substack{i=0 \\ i \neq 2}}^4 T(v_i)$. Thus by (3.2), (3.4) and the

Lemma 3.1, the desired result $N = \sum_{\substack{i=0 \\ i \neq 2}}^4 N_i$ follows.

Case 3. $n_1 = n_3$, $n_2 < n_4$. Similarly (3.3) tells us that if the vectors $v \in T$ have nonzero components of the vectors $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T$, then we must have $v \in v_2$, or $v \in v_3$, or $v \in v_4$. Thus $T = \bigcup_{\substack{i=0 \\ i \neq 1}}^4 T(v_i)$. It follows from

(3.2), (3.4) and the Lemma 3.1 that $N = \sum_{\substack{i=0 \\ i \neq 1}}^4 N_i$.

Case 4. $n_1 < n_3$, $n_2 = n_4$. It follows from (3.3) and the vectors $v \in T$ have nonzero components of the vectors $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T$ that $v \in v_1$, or $v \in v_2$, or $v \in v_4$. So $T = \bigcup_{\substack{i=0 \\ i \neq 3}}^4 T(v_i)$. Then by (3.2), (3.4) and the Lemma 3.1,

one gets that $N = \sum_{\substack{i=0 \\ i \neq 3}}^4 N_i$.

Case 5. $n_1 = n_3, n_2 = n_4$. By the same argument, we know that if the nonzero vectors $v \in T$, then $v \in v_2$, or $v \in v_4$. It infers that $T = \bigcup_{\substack{i=0 \\ i \neq 1,3}}^4 T(v_i)$.

Using (3.2), (3.4) and the Lemma 3.1, we deduce that $N = \sum_{\substack{i=0 \\ i \neq 1,3}}^4 N_i$.

Case 6. $n_1 < n_3, n_2 > n_4$. Similarly, from (3.3) we deduce that if the vectors $v \in T$ have nonzero components of the vectors $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T$, then we obtain $v \in v_1$, or $v \in v_2$, or $v \in v_4$, or $v \in v_5$. Thus $T = \bigcup_{\substack{i=0 \\ i \neq 3}}^5 T(v_i)$.

From (3.2), (3.4) and the Lemma 3.1, the desired result $N = \sum_{\substack{i=0 \\ i \neq 3}}^5 N_i$ follows immediately.

Case 7. $n_1 = n_3, n_2 > n_4$. It's easy to deduce that if the vectors $v \in T$ have nonzero components of the vectors $(u_{11}, \dots, u_{1r_2}, u_{21}, \dots, u_{2r_4}) \in T$, then $v \in v_2$, or $v \in v_4$, or $v \in v_5$. Then $T = \bigcup_{\substack{i=0 \\ i \neq 1,3}}^5 T(v_i)$. Using (3.2), (3.4) and the Lemma

3.1, we have $N = \sum_{\substack{i=0 \\ i \neq 1,3}}^5 N_i$ as desired.

Case 8. $n_3 > n_2$. From (3.3), one knows that if the nonzero vectors $v \in T$, then $v \in v_1$, or $v \in v_3$, or $v \in v_4$, or $v \in v_6$. So $T = \bigcup_{\substack{i=0 \\ i \neq 2,5}}^6 T(v_i)$. Then by (3.2),

(3.4) and the Lemma 3.1, the desired result $N = \sum_{\substack{i=0 \\ i \neq 2,5}}^6 N_i$ follows.

Therefore Theorem 1.3 follows immediately. This ends the proof of Theorem 1.3. \square

4 An example

In this section, we supply an example to illustrate the validity of our main result.

Example 4.1. We use Theorem 1.3 to compute the number N of rational points on the following variety over \mathbb{F}_7 :

$$\begin{cases} x_1 x_2^4 + x_1 x_2^5 + x_1^2 x_2^3 x_3 x_4^4 - 2 = 0, \\ x_1 x_2^5 x_3^3 + x_1 x_2^3 x_3^2 + x_1^2 x_2^4 x_3^3 x_4^5 x_5 x_6 - 4 = 0. \end{cases}$$

Obviously, we have $n_1 = 2, n_2 = 4, n_3 = 3, n_4 = 6$,

$$E^{(1)} = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}, E^{(2)} = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 2 & 3 & 1 & 4 \\ 1 & 5 & 3 & 0 \\ 1 & 3 & 2 & 0 \end{pmatrix},$$

$$E^{(3)} = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 5 & 0 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and } E^{(4)} = \begin{pmatrix} 1 & 4 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 4 & 0 & 0 \\ 1 & 5 & 3 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 & 0 \\ 2 & 4 & 3 & 5 & 1 & 1 \end{pmatrix}.$$

It then follows from the assumption $n_1 < n_3 < n_2 < n_4$, $2, 4 \in F_7^*$ and Theorem 1.3 that

$$N = \sum_{i=2}^4 N_i. \quad (4.1)$$

We first calculate N_2 . Using elementary transformations, One can easily deduce that

$$U^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & -1 \\ -6 & 5 & -2 & 3 \end{pmatrix} \text{ and } V^{(3)} = \begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that

$$U^{(3)} E^{(3)} V^{(3)} = \text{SNF}(E^{(3)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $d_1^{(3)} = d_2^{(3)} = d_3^{(3)} = 1$ and $s_3 = 3$. But Lemma 2.4 tells us that the number of the vectors $(u_{11}, u_{12}, u_{21}, u_{22}) \in (\mathbb{F}_7^*)^4$ satisfying that

$$\begin{cases} u_{11} + u_{12} = 2 \\ u_{21} + u_{22} = 4 \end{cases}$$

is 25. For example, the vector

$$(u_{11}, u_{12}, u_{21}, u_{22}) = (3, 6, 1, 3)$$

is one of the solutions. Choose the primitive element 3 of \mathbb{F}_7^* . Then

$$\begin{aligned} (h'_1, h'_2, h'_3, h'_4)^T &= U^{(3)}(\text{ind}_3 3, \text{ind}_3 6, \text{ind}_3 1, \text{ind}_3 3)^T \\ &= U^{(3)}(1, 3, 6, 1)^T \equiv (1, 2, 1, 0)^T \pmod{6}. \end{aligned}$$

We deduce that the condition $(\widetilde{SE}^{(3)})$ that $1|1$, $1|2$ and $6|0$ holds. Similarly, using Matlab we compute that $H_3 = 4$. So

$$\begin{aligned} N_2 &= q^{\max\{n_2, n_4\} - \min\{n_2, n_4\}} (q^{\min\{n_2, n_4\} - n_3} - (q-1)^{\min\{n_2, n_4\} - n_3}) L_3 \\ &= q^2 (q - (q-1)) H_3 (q-1)^{n_3 - s_3} \prod_{j=1}^{s_3} \gcd(q-1, d_j^{(3)}) = 196. \end{aligned}$$

Consequently, we turn our attention to the computation of N_3 . Using the elementary transformations, one gets that

$$U^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -7 & 5 & 1 & 0 & 0 \\ -9 & 7 & 1 & -1 & 1 \\ -6 & 5 & 0 & -2 & 3 \end{pmatrix} \text{ and } V^{(2)} = \begin{pmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

such that

$$U^{(2)}E^{(2)}V^{(2)} = \text{SNF}(E^{(2)}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $d_1^{(2)} = d_2^{(2)} = d_3^{(2)} = 1$, $d_4^{(2)} = 4$ and $s_2 = 4$. One can easily deduce that the number of the vectors $(u_{11}, u_{12}, u_{13}, u_{21}, u_{22}) \in (\mathbb{F}_7^*)^5$ such that

$$\begin{cases} u_{11} + u_{12} + u_{13} = 2 \\ u_{21} + u_{22} = 4 \end{cases}$$

is 155. Choose the primitive element 3 of \mathbb{F}_7^* . The argument for calculating H_3 and using Matlab we compute that $H_2 = 9$. Hence

$$\begin{aligned} N_3 &= (q^{n_4-n_2} - (q-1)^{n_4-n_2})L_2 \\ &= (q^{n_4-n_2} - (q-1)^{n_4-n_2})H_2(q-1)^{n_2-s_2} \prod_{j=1}^{s_2} \gcd(q-1, d_j^{(2)}) \\ &= 18(q^2 - (q-1)^2) = 234. \end{aligned}$$

Let us now calculate N_4 . Using the elementary transformations, we obtain that

$$U^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -7 & 5 & 1 & 0 & 0 & 0 \\ 15 & -11 & -3 & 0 & 0 & 1 \\ -9 & 7 & 1 & -1 & 1 & 0 \\ -6 & 5 & 0 & -2 & 3 & 0 \end{pmatrix}$$

and

$$V^{(4)} = \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

such that

$$U^{(4)}E^{(4)}V^{(4)} = \text{SNF}(E^{(4)}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $d_1^{(4)} = d_2^{(4)} = d_3^{(4)} = d_4^{(4)} = 1$, $d_5^{(4)} = 4$ and $s_4 = 5$. So Lemma 2.4 gives us that the number of the vectors $(u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}) \in (\mathbb{F}_7^*)^6$ such that

$$\begin{cases} u_{11} + u_{12} + u_{13} = 2 \\ u_{21} + u_{22} + u_{23} = 4 \end{cases}$$

is equal to 961. Choose the primitive element 3 of \mathbb{F}_7^* . By the argument for calculating H_3 and using Matlab, we compute that $H_4 = 84$. Thus one has

$$\begin{aligned} N_4 = l_4 &= H_4(q-1)^{n_4-s_4} \prod_{j=1}^{s_4} \gcd(q-1, d_j^{(4)}) \\ &= 84 \times 6 \times 2 = 1008. \end{aligned}$$

Finally, by (4.1), we have

$$N = \sum_{i=2}^4 N_i = 196 + 234 + 1008 = 1438.$$

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